# One Bald Man ... Two Bald Men ... *Three* Bald Men — Aahh Aahh Aahh Aahh Aaaahhhh!

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*Abstract*

In the context of classical (crisp, precise) sets, there is a familiar connection between the notions of counting, ordering and cardinality. When it comes to vague collections, the connection has not been kept in central focus: there have been numerous proposals regarding the cardinality of vague collections, but these proposals have tended to be discussed in isolation from issues of counting and ordering. My main concern in this paper is to draw focus back onto the connection between these notions. I propose a natural generalisation to the vague case of the familiar process of counting precise collections. I then discuss the relationships between this process of counting and various notions of ordering and cardinality for vague sets. Some existing views concerning the cardinality of vague collections fit better than others with my proposal about how to count the members of such a collection. In particular, the idea that we should approach cardinality via certain formulas of a logical language—which has been prominent in the recent literature—is less attractive than other existing proposals.

*Keywords*

vagueness, fuzzy set, counting, ordinal, cardinality

## 1 Introduction

There is a familiar connection between *counting*, *ordering* and *cardinality*. When we have counted the elements of a collection—let's say, for the sake of example, a collection of brides, brothers, dwarves, or wonders of the world—one, two, three, four, five, six, seven—we have achieved *two* things. First, we have *ordered* the collection: we have put its elements into an ordering from first through to seventh (viz., the order in which we counted them). Second, we have determined *how many* things there are in the collection—that is, the cardinality of the collection: this is given (when we count in the standard way, as in the example above) by the last number we state (in this case, seven).

In sum, when we have a (finite) set or collection of objects, there is a process we can perform on the (elements of the) set: counting. When we have performed this process, we get two things: an ordering of the elements of the set, and an answer to the question how many things are in the set.

This connection between counting, ordering and cardinality is standard fare[1] in the case of classical or 'crisp' collections of objects—collections where there is never any vagueness or indeterminacy regarding whether a given object is in a given collection. When it comes to vague collections, however, the connection has not been kept in central focus in the literature. There have been numerous proposals for answering the question as to *how many* objects there are in a vague collection—that is, what is its cardinality—but these proposals concerning cardinality have tended to be discussed in isolation from the issues of counting and ordering.

In this paper, rather than focussing directly on the cardinality question for vaguely defined collections, I want to begin with the question of how to count vague collections. The aim will be to find a natural generalisation to the vague case of the familiar process of counting precise collections, which then—as in the classical case—yields both a notion of ordering and a notion of cardinality for vague collections. I shall not be proposing any new notions of cardinality for vague collections. What we shall see, however, is that only some of the existing notions mesh nicely with the conception of counting to be introduced here. I take it that potential for coherence with an overall package of concepts analogous to the familiar classical package—counting, ordering and cardinality—is a mark in favour of a given notion of cardinality.

The paper proceeds as follows. §2 reviews the standard set-theoretic reconstruction of the classical story—outlined above in an informal way—of

[1] For example, it is reviewed on the first page of a recent handbook article on set theory [Bagaria, 2008, 616].

counting, ordering and cardinality. §3 introduces vaguely defined collections. In §4, I tell an informal story about counting vague collections, which is intended to generalise the classical story; in §5 I reconstruct this story in set-theoretic terms. In §6 I examine how this picture of counting fits with possible notions of *ordering* vague collections. In §7 I turn to cardinality: the various subsections of §7 look at existing proposals concerning the cardinality of vague collections and explore whether these proposals fit nicely with the story about counting presented in §§4–5.[2]

## 2 Ordinals and Cardinals

Note that in the standard story of the connection between counting, ordering and cardinality, the numbers we recite when we count—one, two, three...—play two different roles: they can function as *ordinals*, which specify the position in an ordering of the objects to which they are assigned (first, second, third,...); and they can function as *cardinals*, which specify how many things there are in a collection (one, two, three,...).

The familiar story is standardly made more precise in the following way. Consider the following sequence of sets, where the first set is the empty set $\emptyset$ and each subsequent set is the set containing all the earlier members of the sequence:

$$\emptyset$$
$$\{\emptyset\}$$
$$\{\emptyset, \{\emptyset\}\}$$
$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$
$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$$
$$\vdots$$

(A piece of terminology that we shall use later: $\omega$ is the infinite set containing all, and only, the sets in the sequence just given.) Following von Neumann, the natural numbers $0, 1, 2, \ldots$ can be identified with the objects (sets) in this sequence: 0 is $\emptyset$, 1 is $\{\emptyset\}$, and so on:

[2] A word of explanation concerning my title: it is a reference to the Count, a character from the television show *Sesame Street*. He loved to count things—and when he had finished doing so, would laugh maniacally (Aahh Aahh Aahh Aahh Aaaahhhh!) to the accompaniment of thunder and lightning.

$$
\begin{aligned}
0 &: \quad \emptyset \\
1 &: \quad \{\emptyset\} \\
2 &: \quad \{\emptyset, \{\emptyset\}\} \\
3 &: \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \\
4 &: \quad \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} \\
&\vdots
\end{aligned}
$$

Note that we can then also write:

$$
\begin{aligned}
0 &: \quad \emptyset \\
1 &: \quad \{0\} \\
2 &: \quad \{0, 1\} \\
3 &: \quad \{0, 1, 2\} \\
4 &: \quad \{0, 1, 2, 3\} \\
&\vdots
\end{aligned}
$$

It can now be seen clearly that the familiar ordering relation $<$ on the natural numbers simply becomes the membership relation $\in$.

We now have the objects that we use for counting (i.e. that we recite, in order, when we count): the natural numbers. Counting itself proceeds as follows. Informally, when we count a collection, we consider (point to, touch) its members in turn, without missing any and without repeating any. Each time we consider an object, we say a natural number, beginning with 1 and then proceeding in order: 2, 3, etc. The formal analogue of counting is a bijection between the set being counted and one of the natural numbers defined above.[3] For example, suppose we are counting dwarves: one, two, three, four, five, six, seven. The analogue of this is a bijection between the set of dwarves and the number 7, that is, the set $\{0, 1, 2, 3, 4, 5, 6\}$. The fact that we count every dwarf (missing none) corresponds to the function from the set of dwarves to 7 being total (or if we are thinking of the function as being from 7 to the set of dwarves, it corresponds to the fact that it is onto);

[3] A function $f : S \to T$ is said to be *total* if it satisfies the condition that every member of $S$ gets sent to some member of $T$; *onto* (aka surjective, a surjection) if it satisfies the condition that every member of $T$ gets hit at least once; and *one-one* (aka one-to-one, into, injective, an injection) if no member of $T$ gets hit more than once. A *bijection* (aka correspondence) is a function that is total, onto and one-one. If there is a bijection $f$ from $S$ to $T$, then there is a bijection (the inverse of $f$) from $T$ to $S$; hence it is common to talk non-specifically of a bijection *between* $S$ and $T$.

the fact that we do not count any dwarf more than once corresponds to its being a function (or if we are thinking of the function as being from 7 to the set of dwarves, it corresponds to the fact that it is one-one).

Informally, counting yields an ordering of the set being counted, and a cardinality for that set. In the formal reconstruction, this comes out as follows. If there is a bijection between the set of dwarves and the number 7, then that number *just is* the cardinal number of that set. As for ordering, the natural numbers come in a standard, familiar order: $0, 1, 2, \ldots$. As we have remarked, their formal analogues also come in a corresponding order, given by the set membership relation. Now suppose we consider each number not simply as a set—as we do when we think of it as a cardinal number—but as an ordered set: a set together with the ordering relation given by $\in$. Then, given a bijection between a number and a set, we get a corresponding ordering of that set. When we think of our numbers in this way—as ordered sets—they become ordinals.

Note the difference between a particular *ordering* of a set, and its corresponding *ordinal*. There are many different ways of counting the dwarves—first Bashful, then Doc, then Dopey, Grumpy, Happy, Sleepy and finally Sneezy; or Grumpy first, then Sleepy, Sneezy, Doc, Dopey, Happy and then Bashful last; etc. Each of these is represented by a *different* bijection between the set of dwarves and the number 7. But when we abstract away from the particular identities of the objects in the ordering, and just look at the *type* of ordering we get, we see that we get the same type of ordering each time: one object, then another, then another, then another, then another, then another and finally another—seven things in a row. An *ordinal* represents an order *type*. So the multiple different orderings of the set of dwarves all correspond to the same ordinal, 7.[4]

Note also that in the informal story, 0 plays no role—whereas in the set-theoretic reconstruction, it does. In the informal story, we count $1, 2, 3, \ldots$—starting at 1—and the cardinal number of the set we are counting is the last number stated. In the set-theoretic reconstruction, $n$ is the set $\{0, 1, 2, \ldots, n-1\}$, which has $n$ elements: but not the numbers $1 \ldots n$, rather the numbers $0 \ldots n-1$. The counting process is represented as a bijection between the set being counted and a number $n$. The bijection associates the first element in the set (i.e. first in the ordering generated by the counting process) with 0 (not with 1) and the last with $n-1$ (not with $n$). The cardinal number

[4]An ordinal, as Cantor [1915] put it, results from a single act of abstraction: we ignore the particular identity of each object in the set and simply look at the order in which these objects appear; a cardinal results from a double act of abstraction, in which we ignore both the particular identity of each object in the set and the order in which these objects appear, paying attention only to the number of objects in the set.

of the set is then the set of all the numbers associated with objects in the set—which is the number after the last one associated with an object in the set, rather than the last one itself. One might therefore think that if we wish to speak strictly correctly, we need to say—for example—that the set-theoretic story reconstructs, not the standard counting procedure itself, but an equally good alternative procedure that starts from 0 (instead of 1) and assigns as cardinal the first number not stated (rather than the last number stated). We shall not enter into these sorts of issues here, as they would be a distraction in the present context. For our purposes it will be best to speak simply of the set-theoretic story as a reconstruction of the familiar counting process (which starts from 1)—leaving it to readers who regard any of our formulations as strictly speaking incorrect to re-word them mentally to their own satisfaction.

Summing up: In the formal version of the familiar story, we have a sequence of sets. They play the role of numbers. If we think of them simply as sets, they are cardinal numbers; if we think of them furthermore as ordered (by set membership) they become ordinal numbers. Counting a set and getting the answer $n$ corresponds to the existence of a bijection between that set and the number $n$. Such a bijection yields two things: an ordering of the set (transferred from the ordering of $n$, when we consider it as an ordinal) and an answer to the question how many things are in that set ($n$ itself, when we think of it as a cardinal).

The classical story just told extends from crisp finite collections to crisp infinite collections in a standard way (as explained in any introductory work on set theory). In this paper we wish to generalise in a different direction: we shall consider only finite sets—but sets whose membership is not precisely defined.

## 3 Vaguely Defined Collections

Given a predicate $P$, we can (try to) count the set of $P$'s. How to proceed, when $P$ is vague? For example, suppose that there are twenty men in the room: ten professional basketball players, six professional jockeys, and four more or less borderline cases of tallness. How to count the tall men in the room? Obviously we count each of the basketball players, and none of the jockeys—but what about the borderline tall men? It is unclear whether we should count them or not.

A similar problem arises if we suppose that the identity relation can be vague.[5] For example, suppose that Jane, who is $5'8''$, and Emma, who is

[5] See for example Parsons [2000, §8.1].

$6'1''$, work at moon base 9. One morning, Emma teletransports to base 7 for the day, returning that evening to base 9. Suppose we wish to count the persons of $6'$ or more in height who were present in base 9 that day. '$6'$ or more in height' is a precise predicate—and yet we face a similar problem to the one we face when we wish to count the tall men: it is unclear whether the Emma who steps out of the teletransporter in the evening is a distinct person from the Emma who stepped into the teletransporter in the morning; hence, having counted Emma in the morning, it is unclear whether (in addition) to count Emma in the evening.[6]

Of course, if there can be cases of the two sorts just described, then there can also be hybrid cases, involving both vague identity and vague predicates—for example, counting the tall persons in base 9 on a given day.

I have argued elsewhere that ultimately sense cannot be made of the idea of vague identity [Smith, 2008a]. I shall therefore focus entirely on cases of counting collections whose vagueness arises from the vagueness of some predicate used to define the set: for example, the tall men, the bald men, the heavy suitcases, the long walks, and so on. I have also argued elsewhere that vague predicates should be analysed in terms of degrees of truth—in particular, using fuzzy sets [Smith, 2008b]. I shall therefore carry out my discussion of vaguely defined collections in terms of fuzzy sets. Nevertheless, much of what I say could be applied, mutatis mutandis, both to other approaches to vagueness, and to counting issues arising from vague identity. Therefore, I shall often talk generally of 'vague sets', 'vague collections' and so on, rather than specifically of 'fuzzy sets'—even though at all points at which rigour is required, the precise technical development will be in terms of fuzzy sets. This paper is intended as a general contribution to the literature on counting and cardinality in the presence of vagueness, illustrated in terms of one particular source of vagueness (vague predication, not vague identity) modelled in one particular way (using fuzzy sets). Some readers may take vague identity seriously, or they may model vague predication using machinery other than fuzzy sets: most of what I say should still be relevant to such readers.

Some terminology: $[0, 1]$ is the closed real unit interval, comprising all the real numbers between 0 and 1 inclusive: that is, all real numbers $x$ with $0 \leq x \leq 1$. $(0, 1]$ is the set of all real numbers $x$ with $0 < x \leq 1$, and $[0, 1)$ is the set of all real numbers $x$ with $0 \leq x < 1$. A fuzzy subset $S$ of some background set $M$ is a function from $M$ to $[0, 1]$; the number assigned to

[6] The teletransporter is playing the role of disrupter. Readers who do not like the example should substitute their favourite case from the personal identity literature of a disruptive process where it is unclear whether the person who enters the process is the same as the person who exits the process.

$x \in M$ represents $x$'s degree of membership in $S$. The set of all things in $M$ assigned a value strictly greater than 0 is called the *support* of $S$, here denoted $S_*$. The set of all things in $M$ assigned the value 1 is called the *core* (or kernel) of $S$, here denoted $S^*$. Note that the support and the core of a fuzzy set are both crisp sets.

Throughout—as already foreshadowed—we shall restrict our attention to fuzzy sets whose support is a finite set. (Note that the background set $M$ need not be finite.) In the classical case, things get really interesting when we move beyond the realm of finite sets; as we shall see, in the vague case things are already rather interesting in the finite case.

## 4 Counting: The Informal Story

Suppose then that we wish to count, say, the tall men in the room. Of course we cannot simply count them in the usual way: the familiar procedure of intoning $1, 2, 3, \ldots$ as we go through the members of the set—being sure not to miss any nor to count any twice—simply 'crashes' if we get to an object such that it is unclear whether or not that object is in the set. If it is in, we count it; if it is out, we do not—but the familiar procedure assumes everything is in or out, and hence it breaks down when we confront a set with elements that are to some degree in and to some degree out. Of course we can count—in the familiar way—any precise sets in the vicinity: the set of men who are greater than $6'$ in height; the set of men who are members of the set of tall men to a degree greater than 0.5; and so on. But the issue here is whether we can go further. Can we generalise the familiar procedure of counting the members of a crisp set, to the case of vague sets?

I can think of only one natural, satisfactory way of extending the usual counting procedure. In the classical procedure, we intone the counting numbers in turn: $1, 2, 3, \ldots$. We assign one number to each object that is in the set—and no number to any object that is not in the set. Thus, the tagging of objects with numbers is an on/off matter: objects that are in the set get tagged with a number and those that are not in do not get tagged. The degree of tagging—the strength of the glue with which the tag is affixed to the object, so to speak—matches the degree of membership of the object tagged in the set being counted: it is 'full on' or 'full off'.

In the new context of vague sets, objects can be completely in a set (in it to degree 1) and they can be completely out of a set (in it to degree 0), and objects can also be in a set to any intermediate degree. Maintaining the idea that the degree of tagging of an object should match the degree of membership of the object tagged in the set being counted, we now tag objects to various degrees. That is, we attach numbers to objects—but some are attached more firmly than others. So, the counting procedure is this. Go through the members of the set to be counted, intoning the counting numbers in turn: $1, 2, 3, \ldots$. For each object we come to, the degree of attachment of the tag (i.e. counting number) to the object matches the degree of membership of that object in the set being counted. We can think of this degree of attachment as being expressed by confidence—or loudness, or what have you—of intonation. If we come to an object that is fully in the set, we intone the next number with full confidence; if we come to an object that is not in the set at all, we do not intone anything (we save the next number for the next object that is in the set to some non-zero degree); if we come to an object that is in the set to an intermediate degree, we intone the next counting number with a degree of hesitation—or at a volume, or whatever—that matches the degree of membership of that object in the set being counted.

For example, suppose that Allison, Bridget, Caroline, Diana, Eleanor, Frances, Greta and Hazel (and no-one else) are in a room. Suppose that their degrees of membership in the set of tall persons in the room are as follows (where $x/y$ denotes the degree $x$ of membership of the person with initial $y$):

$$1/a, \quad 0.5/b, \quad 0.8/c, \quad 1/d, \quad 0/e, \quad 0.2/f, \quad 0.9/g, \quad 0.3/h$$

Then we might count the members of the set of tall persons in the room as follows (where the table is to be read this way: looking at the person named in the left column, we intone the number in the middle column with the degree of hesitation given in the right column; or this way: to the person named in the left column, we attach—with the degree of attachment given in the right column—the number given in the middle column):

| | | |
|---|---|---|
| Allison | 1 | 1 |
| Bridget | 2 | 0.5 |
| Caroline | 3 | 0.8 |
| Diana | 4 | 1 |
| Frances | 5 | 0.2 |
| Greta | 6 | 0.9 |
| Hazel | 7 | 0.3 |

Note that Eleanor does not get assigned any number—not even to a tiny degree—because her degree of membership in the set being counted is 0.

Here's another representation of this counting process, where this time the strength of attachment of number to person is indicated by the density of the type in which the number is written (with the idea being that 1 is written in 100% black ink, 2 in 50% greyscale, 3 in 80% greyscale, and so on down to 7 in 30% greyscale):

| | |
|---|---|
| Allison | 1 |
| Bridget | 2 |
| Caroline | 3 |
| Diana | 4 |
| Frances | 5 |
| Greta | 6 |
| Hazel | 7 |

Of course, there is no reason why we should count the members of the set in the order we just did. An equally good way of counting would be the following:

| | | |
|---|---|---|
| Bridget | 1 | 0.5 |
| Caroline | 2 | 0.8 |
| Allison | 3 | 1 |
| Diana | 4 | 1 |
| Greta | 5 | 0.9 |
| Hazel | 6 | 0.3 |
| Frances | 7 | 0.2 |

| | |
|---|---|
| Bridget | 1 |
| Caroline | 2 |
| Allison | 3 |
| Diana | 4 |
| Greta | 5 |
| Hazel | 6 |
| Frances | 7 |

As would the following:

| | | |
|---|---|---|
| Diana | 1 | 1 |
| Hazel | 2 | 0.3 |
| Frances | 3 | 0.2 |
| Bridget | 4 | 0.5 |
| Caroline | 5 | 0.8 |
| Greta | 6 | 0.9 |
| Allison | 7 | 1 |

| | |
|---|---|
| Diana | 1 |
| Hazel | 2 |
| Frances | 3 |
| Bridget | 4 |
| Caroline | 5 |
| Greta | 6 |
| Allison | 7 |

And so on. Note that for each element of the set, the number assigned to that element need not remain the same across different ways of counting the set—but the degree to which its number (whatever number it is) is assigned does remain the same: it corresponds to the degree of membership of that element in the set.[7]

[7] I said that much in this paper could be applied, mutatis mutandis, both to approaches to vagueness that do not employ fuzzy sets and to counting issues arising from vague identity. The story that I have just told about counting vague collections extends in an obvious way to any treatment of vagueness wherein the extension of a vague predicate can

## 5 Counting: The Formal Reconstruction

In the set-theoretic reconstruction of the standard picture of counting a crisp finite set, the counting process is represented by a bijection between the set $S$ being counted and a natural number—which is seen as a set of objects. The standard ordering on this natural number (i.e. on the elements of the set with which this number is identified) yields (via the bijection) an ordering of the set being counted. The natural number—together with the standard ordering of its elements—plays the role of an ordinal (denoted $\bar{S}$). It also—when considered by itself, without the ordering of its elements—plays the role of a cardinal (denoted $\bar{\bar{S}}$).[8]

We want to follow a similar line of thought in relation to vague collections. In the previous section we told a story about counting the members of a vaguely defined set. Our first task now is to give a more precise reconstruction of this story in set-theoretic terms.

The process of counting the members of a fuzzy set $S$ can be represented by a function from $S_*$ to $(0, 1] \times \omega$ satisfying the following conditions:[9]

1. the function is total

   (Everything is counted.)

2. the function is one-one

   (Two different things are never conflated and counted as one.)

3. each element of $\bar{\bar{S}}_*$ appears *exactly once* in the image of the function[10]

   (The image of the function, for a given set $S_*$, is a set of pairs; the idea here is that if we look at all the second elements of these pairs, each element of $\bar{\bar{S}}_*$ appears exactly once. This captures the idea that we go through the elements of the support of $S$ one by one, assigning

be modelled as a function from the domain of discourse to a set of membership values—for example, supervaluationist (or subvaluationist) treatments and treatments employing a many-valued or gappy (or glutty) logic. For the case of vague identity, the extent to which the next counting number is attached to the next object in the set should reflect both the extent to which that object is a member of the set and the extent to which it is distinct from all other objects in the set.

[8] The notation is Cantor's. Each bar represents an act of abstraction: one for an ordinal, two for a cardinal (see n.4 above).

[9] The symbol $\times$ represents the Cartesian product. $S \times T$ is the set of all ordered pairs whose first element is a member of the set $S$ and whose second element is a member of the set $T$.

[10] Recall that $\bar{\bar{S}}_*$ is the number of elements in the support of $S$—and we may think of this number as a set.

successive numbers to them—just as in the classical story; the only difference, which we get to below, is that the association of each number is now a matter of degree.)

4. for each object $x$ in $S_*$, the first element of the pair to which $x$ is mapped by the function is the same as $x$'s degree of membership in $S$

   (This captures the idea that as we count the elements of the support of $S$, the degree to which we associate the next counting number with the next object considered is the same as that object's degree of membership in $S$.)

This function assigns to each $x \in S_*$ a pair of things: the second element in the pair is a counting number (of the ordinary classical sort); the first element represents the degree to which that counting number is attached to $x$.

Let's refer to each member of $(0, 1] \times \omega$—i.e. each pair $(x, n)$ whose first element $x$ is a real in $(0, 1]$ and whose second element $n$ is a natural number—as a *weighted* number, or more specifically a weighted version of the number $n$. We can then describe the present proposal as follows: we represent the process of counting a vague set as a function that assigns to each element of the support of that set a weighted version of one of the numbers $1 \ldots n$, where $n$ is the number of elements in the support; furthermore, the function assigns these numbers in such a way that a weighted version of each of the numbers $1 \ldots n$ gets assigned to some element of the support, no two elements get assigned a weighted version of the same number, and the weighting on $n$ in the weighted number assigned to $a$ is precisely the degree of membership of $a$ in $S$.

If we look back at the tables in the previous section, we can now see them as pictures of counting functions of the sort just described.

## 6 Ordering

In the classical story, the process of counting the objects in a set yields an ordering of the set: the order in which we count the elements. In the set-theoretic reconstruction, the counting process is represented by a bijection between the set $S$ being counted and a natural number. This natural number—thought of as a set—comes with a natural ordering. This ordering then yields—via the bijection—an ordering of the set being counted.

Can we tell a similar story in the vague case? We have represented the process of counting a vague set $S$ as a function which assigns to each element of the support of $S$ a pair. The first element of the pair is a real number;

the second is a natural number. If there is a natural way of ordering these pairs, it will yield (via the counting function—which is total and one-one) an ordering of $S_*$.

It seems to me that there are two natural orderings on the pairs. (This is typical: where, in the classical case, there is one natural option, there are usually multiple equally natural options when we move to the fuzzy case.) The first ordering puts $(x_1, y_1) < (x_2, y_2)$ iff $y_1 < y_2$. (The $y$'s are natural numbers, and the most recent occurrence of $<$ denotes the standard ordering on the natural numbers.) The resulting ordering of $S_*$ is the one that simply takes the members in the order we count them—ignoring any differences in the degrees to which successive counting numbers are attached to these objects. (Note that the ordering of the pairs ignores the $x$'s altogether.)

The second ordering puts $(x_1, y_1) < (x_2, y_2)$ iff either $x_1 > x_2$, or $x_1 = x_2$ and $y_1 < y_2$. The resulting ordering of $S_*$ is the one that goes through the members in order of degree of membership—starting with the degree 1 members, if there are any, and then working down. Where there are multiple elements with the same degree of membership, they are ordered in the order in which they were counted.[11]

Both of these options result in a crisp, linear ordering of $S_*$. The order types of these orderings are simply classical ordinals—and just as in the classical (finite) case, the order in which we count the elements of a set does not affect the resulting ordinal. No matter what order we count it in, and no matter whether we take the first or the second option just discussed, the ordinal associated with a fuzzy set will simply be the classical ordinal associated with its support. In other words, there are different options regarding the order in which we put the objects in the set—but the resulting order type will, in the cases discussed so far, simply be '$n$ objects in a row', where $n$ is the number of objects in the support.

Another kind of possibility would be to look for a fuzzy ordering of $S_*$: that is, a mapping from $S_* \times S_*$ to $[0, 1]$ (rather than to $\{0, 1\}$, as in the case of a crisp ordering). Presumably one would want the degree to which $x$ comes before $y$ to be a function of both $x$'s and $y$'s degrees of membership in $S$ and the order in which they were counted—that is, a function of both which counting numbers are assigned to objects when we count the set and the strengths of those assignments. We shall not explore the options here any further in this paper. Suffice it to note that orderings of this kind could be derived from the process of counting a vague set, modelled in the way suggested here—and our concern is to preserve the *connection* between the

[11] Of course there is also a reverse version of this ordering, where we begin with the lowest degree members and work up.

notions of counting and ordering, rather than to explore the options regarding ordering in detail.

## 7 Cardinality

There are numerous options in the literature regarding the notion of the cardinality of a vague set—that is, numerous proposals for how to answer the question as to *how many* things there are in a vague set. Our concern here is that the answer to the cardinality question should flow from the output of the counting process: once we have counted a vague set, we should have sufficient resources in hand to answer the 'how many?' question.

This is how things go in the classical case. In the informal version of the story, the counting process consists in tagging each object in the set with a number. The output of this process is a list of numbers: $1 \dots n$. The cardinality of the set is then the last of these numbers. In the set-theoretic reconstruction, the counting process is represented by a (bijective) function between the set $S$ being counted and some natural number $n$ (thought of as a set). The output of this process—the image of the set being counted under this function—is a set of numbers/sets $0 \dots n-1$. This set of numbers—which is itself the number $n$—is then the cardinality of the set being counted.

Turning to the vague case, in the informal version of the story, the counting process consists in tagging each object in the set with a number—with the strength of attachment of the tag matching the level of membership of the object being tagged in the set being counted. The output of this process is a list of numbers, $1 \dots n$, with each number said in a softer or louder voice—or written in a lighter or darker shade of grey. In the set-theoretic reconstruction, the counting process is represented by a function (satisfying certain constraints) from the support $S_*$ of the fuzzy set $S$ being counted to pairs of reals in $(0, 1]$ and natural numbers. The output of this process—the image of the support under this function—is a set of pairs of reals in $(0, 1]$ and natural numbers. The idea now is that we should be able to derive the cardinality of $S$ from this set of pairs—from the set of pairs that we get as output when we count $S$. We should not have to return to $S$ itself, nor draw on any other sources of information. Just looking at the list of numbers, written in varying shades of grey, should be enough to answer the question as to how many objects there are in the fuzzy set.

Here is a straightforward idea. The cardinality of a crisp set $S$ is simply the set that gathers together the values of the counting function that we get when we take members of $S$ as input: $0, 1, 2, \dots, n-1$ for some $n$. Now when we count a fuzzy set $S$, the values of the counting function that we get when we take members of $S_*$ as input are pairs: $(x_0, 0)$, $(x_1, 1)$, $(x_2, 2)$, $\dots$,

$(x_{n-1}, n-1)$ for some $n$, where the $x_i$'s are reals in $(0,1]$. Such a set of pairs determines a fuzzy subset of $n$ (where $n$ is conceived as the set containing $0, \ldots, n-1$): the fuzzy subset that assigns as degree of membership to each member of $n$, the number with which it is paired in the list of outputs. So: can we not take this fuzzy subset of $n$ to be the cardinality of $S$?

We cannot: because if we count $S$ again in a different order, we will (in general) get a *different* fuzzy subset of (the same natural number) $n$. (If, on one way of counting, 1 is assigned to a degree 0.8 member of $S$, then 1 will be a degree 0.8 member of the resulting fuzzy subset of $n$; if, on another way of counting, 1 is assigned to a degree 0.3 member of $S$, then 1 will be a degree 0.3 member of the resulting fuzzy subset of $n$; and so on.) Yet it is a fundamental constraint on the notion of cardinality that simply changing the *order* in which we count the elements of a set should not change the answer we get as to *how many* objects there are in the set.[12]

At this point, rather than trying to make up new proposals regarding the cardinality of vague collections, we shall turn to the numerous proposals already in the literature, and ask whether these proposals fit with the account of counting given above. We shall not consider every proposal that has been made; rather, we shall consider some proposals that play a prominent role in the current literature on this topic.[13]

### 7.1 Cardinalities as Natural Numbers

The first class of proposals holds that the *form* of the answer to the question 'How many objects are in the set?' should be a natural number—in the vague case as well as the classical case. The natural proposals in this area are as follows. The cardinality of a fuzzy set $S$ is the (classical) cardinality of:

1. the support of $S$
2. the core of $S$
3. $S^x$, where $S^x$ is the (crisp) set of all elements whose degree of membership in $S$ is strictly greater than $x$, for some specified threshold $x \in [0,1)$
4. $S_x$, where $S_x$ is the (crisp) set of all elements whose degree of membership in $S$ is greater than or equal to $x$, for some specified threshold $x \in (0,1]$

[12] Recall Cantor's second act of abstraction

[13] My judgements regarding prominence in the literature have been heavily influenced by Wygralak [2003], which readers should consult for further details of—and bibliographical references regarding—the views discussed in §§7.1–7.3.

Obviously, cardinalities of all these sorts can readily be extracted from the output of the process of counting $S$ in the way presented above. To find the cardinality in sense 1, we count up (in the classical way) *all* the weighted numbers in the output of the process of counting the vague set $S$. To find the cardinality in sense 2, we count up the weighted numbers whose weight is 1—that is, the numbers written in 100% black ink. To find the cardinality in sense 3, we count up the weighted numbers whose weight exceeds $x$—that is, the numbers written at a level of greyscale darker than $x$%; and so on.

### 7.2 Cardinalities as Real Numbers

The second class of proposals holds that the form of the answer to the question 'How many objects are in the set?' should be a single number—but a nonnegative real number, not necessarily a nonnegative natural number (as in the classical case). The most natural proposal here is that the cardinality of $S$ is the sum, over all $x$ in the support of $S$, of the degree of membership of $x$ in $S$. This is called the *sigma count* of $S$, denoted $sc(S)$:

$$sc(S) = \sum_{x \in S_*} S(x)$$

(Here, $S(x)$ denotes the degree of membership of $x$ in $S$—i.e. the value assigned to $x$ by $S$, when we think of $S$ as a function from some background set to $[0, 1]$.) So when we are counting up the bald men, a degree 1 bald man adds 1 to the count, a degree 0.3 bald man adds 0.3 to the count, and in general a degree $x$ bald man adds $x$ to the count.[14]

Obviously, the sigma count of $S$ can be extracted from the results of counting the members of $S$ in the way presented above. The output of the counting process is a bunch of weighted numbers; to get the sigma count, we simply add the weights on these numbers.

### 7.3 Cardinalities as Fuzzy Sets of Natural Numbers

The first class of proposals held that the form of the answer to the question 'How many objects are in the set?' should be a natural number—in the vague case as well as the classical case. The second class of proposals generalised in one direction—maintaining that the cardinality should be a single number, but not requiring that it be a natural number. The third class of proposals

[14]Compare the way that universities count students for certain purposes: a full-time student adds 1 to the count; a half-time student adds 0.5 to the count; and so on. (Thanks to David Braddon-Mitchell for this example.)

generalises in a different direction, holding that the cardinality of a fuzzy set should be a fuzzy set of natural numbers, rather than a single such number.

One proposal along these lines is as follows. For each natural number $n$, we ask 'What is the highest level at which we can set the membership threshold $x$, such that the number of things that are in $S$ to a degree of at least $x$ is at least $n$?' The answer—a real in $[0, 1]$—is the degree of membership of $n$ in the fuzzy set of natural numbers that constitutes (on this proposal) the cardinality of $S$. More precisely, the cardinality of $S$ is a fuzzy subset of the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers—that is, a function $l : \mathbb{N} \to [0, 1]$—defined as follows. For each $n \in \mathbb{N}$:

$$l(n) = \sup\{x \in (0, 1] : \bar{\bar{S}}_x \geq n\}$$

Note that if there is no positive threshold $x$ such that at least $n$ things are in $S$ to degree $x$ or more, then $l(n) = 0$.

Recall the example of the fuzzy set of tall persons described in §4, with degrees of membership as follows (where $x/y$ denotes the degree $x$ of membership of person $y$):

$$1/a, \quad 0.5/b, \quad 0.8/c, \quad 1/d, \quad 0/e, \quad 0.2/f, \quad 0.9/g, \quad 0.3/h$$

The cardinality of this fuzzy set—on the present proposal—is the following fuzzy subset of $\mathbb{N}$ (where $x/n$ denotes the degree $x$ of membership of the number $n$):

$$1/0,\ 1/1,\ 1/2,\ 0.9/3,\ 0.8/4,\ 0.5/5,\ 0.3/6,\ 0.2/7,\ 0/8,\ 0/9,\ 0/10,\ \ldots$$

The cardinality of $S$ in this sense is readily recoverable from the output of the process of counting $S$ in the way presented above. The output of the counting process is a bunch of weighted numbers. To get the cardinality, we write out the *weights* in nondecreasing order (including any repetitions)—in the present example:

$$1, \quad 1, \quad 0.9, \quad 0.8, \quad 0.5, \quad 0.3, \quad 0.2$$

(Note that we do not write 0 at the end of this list, because when we count we get a weighted number for each member of the *support* of $S$—i.e. each thing that is a member of $S$ to some non-zero degree.) The cardinality that we seek is a fuzzy subset of $\mathbb{N}$—a function that assigns a degree of membership to each $n \in \mathbb{N}$. For $n = 0$, the degree of membership of $n$ is 1. For $n$ greater than 0, and less than or equal to the number of things in the list of weights that we just wrote out, the degree of membership of $n$ is simply the $n$th weight in the list. For all larger $n$, the degree of membership of $n$ is 0.

The cardinality proposal that we just looked at sees the cardinality of $S$ as a fuzzy subset of $\mathbb{N}$, where the degree of membership of $n$ in this fuzzy subset is a measure of the truth of the claim that there are *at least* $n$ things in $S$. A second proposal replaces 'at least' here with 'at most'. On this proposal, the cardinality of $S$ is a function $m : \mathbb{N} \to [0, 1]$ defined as follows:

$$m(n) = 1 - l(n + 1)$$

A third proposal replaces 'at least' in the first proposal with 'exactly'. On this proposal, the cardinality of $S$ is a function $e : \mathbb{N} \to [0, 1]$ defined as follows:

$$e(n) = \min\{l(n), m(n)\}$$

As cardinality in the sense of $l$ can be extracted from the output of the process of counting a vague set, evidently so can cardinality in the senses of $m$ and $e$.

### 7.4 Cardinalities via Logical Formulas

It is well known that for any finite $n$ and any predicate $P$, there are formulas of first order logic that are true in exactly those (classical) models in which the extension of $P$ contains exactly $n$ things. There are different recipes for constructing such numerical formulas. For example—Recipe 1—we can represent 'There are exactly $n$ $P$'s' as the conjunction of 'There are at least $n$ $P$'s' and 'There are at most $n$ $P$'s', where the 'at least' claims are rendered as follows:

1. $\exists x Px$
2. $\exists x \exists y (Px \wedge Py \wedge x \neq y)$
3. $\exists x \exists y \exists z (Px \wedge Py \wedge Pz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$

$\vdots$

and the 'at most' claims are rendered as follows:

1. $\forall x \forall y ((Px \wedge Py) \to x = y)$
2. $\forall x \forall y \forall z ((Px \wedge Py \wedge Pz) \to (x = y \vee x = z \vee y = z))$
3. $\forall x \forall y \forall z \forall w ((Px \wedge Py \wedge Pz \wedge Pw) \to (x = y \vee x = z \vee x = w \vee y = z \vee y = w \vee z = w))$

$\vdots$

Recipe 2 is just like Recipe 1 except that the 'at most' claims are rendered as follows:

1. $\neg\exists x\exists y(Px \wedge Py \wedge x \neq y)$

2. $\neg\exists x\exists y\exists z(Px \wedge Py \wedge Pz \wedge x \neq y \wedge x \neq z \wedge y \neq z)$

3. $\neg\exists x\exists y\exists z\exists w(Px \wedge Py \wedge Pz \wedge Pw \wedge x \neq y \wedge x \neq z \wedge x \neq w \wedge y \neq z \wedge y \neq w \wedge z \neq w)$

$\vdots$

That is, 'There are at most $n$ $P$'s' is the negation of 'There are at least $n+1$ $P$'s' (as rendered above). Recipe 3 does not represent 'There are exactly $n$ $P$'s' as the conjunction of 'There are at least $n$ $P$'s' and 'There are at most $n$ $P$'s', but simply renders the 'exactly' claims as follows:

1. $\exists x\forall y(Py \leftrightarrow y = x)$

2. $\exists x\exists y(x \neq y \wedge \forall z(Pz \leftrightarrow (z = x \vee z = y)))$

3. $\exists x\exists y\exists z(x \neq y \wedge x \neq z \wedge y \neq z \wedge \forall w(Pw \leftrightarrow (w = x \vee w = y \vee w = z)))$

$\vdots$

There are further options besides these three.[15]

Parsons's approach to the issue of counting in the presence of vagueness is as follows [Parsons, 2000]. If we want to know how many $P$'s there are—where either $P$ is a vague predicate, or indeterminacy of identity is involved, or both—we consider each of the numerical formulas in turn and assess its truth value. So, in our example of the fuzzy set of tall persons with degrees of membership as follows:

$$1/a, \quad 0.5/b, \quad 0.8/c, \quad 1/d, \quad 0/e, \quad 0.2/f, \quad 0.9/g, \quad 0.3/h$$

what we do is consider each numerical formula—'There is exactly one $P$', 'There are exactly two $P$'s', etc.—and determine its degree of truth relative to a model in which $P$ is assigned as extension the fuzzy set just described.[16]

There are two ways of interpreting what is going on here. The first way—which I take to be the sort of thing Parsons has in mind—is that the possible answers to the cardinality question (i.e. 'How many tall persons are there in the room?') are natural numbers. But it may not be that a

[15] For more details on the foregoing material, see e.g. Smith [2012, §13.5].

[16] Parsons does not work with fuzzy sets or degrees of truth. Here and below I adapt his ideas to the present context, in which we use fuzzy sets to model vagueness.

unique answer is correct and all others incorrect. Various answers—various numerical formulas—may each have a non-zero degree of truth.[17] On this interpretation, the present approach does not yield a single object as cardinal number of the set of tall persons: it just yields an assessment (in the form of a degree of truth) of each possible answer. This is unsatisfactory: our goal is to extract a cardinality from the output of the counting process—not to deny that there is such a thing as 'the cardinality' of a vague set. But of course there is a second way of developing the present idea: we take the cardinality of a fuzzy set $S$ to be a fuzzy subset of $\mathbb{N}$: the one that assigns as degree of membership to each $n \in \mathbb{N}$ the degree of truth of the $n$th numerical formula on a model on which $P$ has $S$ as its extension.

Evidently, once we move beyond the classical framework, numerical formulas constructed according to different recipes—which are classically equivalent—need not remain equivalent. Thus, we shall get different versions of the present story—different cardinalities for vague sets—depending on which recipe we pick for constructing our numerical formulas, and depending on the truth conditions that we adopt for the logical operators in the new non-classical setting.

Our concern here is with whether the cardinality of a vague set can be recovered from the output of the process of counting that set. So: if we have counted a vague set $S$, and have to hand the output of the counting process—a list of weighted numbers—can we reconstruct the truth values of the numerical formulas on a model on which the predicate $P$ has the set $S$ as its extension? Note that we do not have the set $S$ itself to hand—we have only the list of weighted numbers. But of course this list allows us to reconstruct how many things are in the support of $S$, and their degrees of membership in $S$—and so, given certain assumptions about how the model theory is supposed to work in the new vague context, we can indeed reconstruct the truth values of the numerical formulas.

But now the question arises: why should we want to go this long way around—via the numerical formulas (and furthermore settling on a particular choice of recipe for constructing them, and a particular set of truth conditions for the logical operators)—rather than simply extracting the desired cardinality (fuzzy set of natural numbers) directly from the output of the counting process? (For example, note that if we define the truth con-

[17]See for example Parsons [2000] p.135: "It follows that the question of how many persons there are all told has no correct answer. . . . in this case it seems clear that these are the right things to say: any answer less than two or more than three is wrong, and either "two" or "three" is such that it is indeterminate whether it is correct" and p.136: "It appears that in this case any answer less than one or more than three is definitely wrong, but the answers "one", "two", or "three" should all have indeterminate truth-value."

ditions for negation and conjunction as follows—where $|\alpha|$ is the degree of truth of the formula $\alpha$:

$$|\neg\alpha| = 1 - |\alpha|$$

$$|\alpha \wedge \beta| = \min\{|\alpha|, |\beta|\}$$

and the truth condition for the existential quantifier in terms of sup, and if we construct our numerical formulas according to Recipe 2, then the cardinality that we arrive at for a given fuzzy set by going via the numerical formulas will turn out to be the same as cardinality in sense $e$ of §7.3.) There is only one possible reason: we might think that this route, while lengthy, is *conceptually* correct. That is, we might think that there is some special relationship between the numerical formulas and questions of cardinality—a connection that it is important to retain. This seems to be what Parsons thinks. He refers to the numerical formulas as *analyses* of cardinality claims and writes "These analyses are natural hypotheses about the meaning of cardinality claims" [Parsons, 2000, 139]. Hyde, who follows Parsons's approach to cardinality issues in the context of vagueness, also refers to the numerical formulas as *analyses* of claims of the form 'There are exactly $n$ $P$s' [Hyde, 2008, 171]. However, I think that this is the wrong attitude to numerical formulas. The fact that for any finite $n$ and any predicate $P$, there are formulas of first order logic that are true in exactly those (classical) models in which the extension of $P$ contains exactly $n$ things, is not properly seen as a fundamental fact about what it means for there to be $n$ $P$'s. It is a fact about the expressive power of (classical) first order logic. It is a useful fact—but if it did not hold, that would not reflect badly on the concept of cardinality: it would reflect badly on the logic. We would still know exactly what it means for there to be $n$ $P$'s—it would just be something that we could not express in a logical formula. Consider the claim 'There are finitely many $P$'s'. It is well known that we cannot construct a formula—or even a set of formulas—such that on every model on which that formula—or all the formulas in that set—are true, the extension of $P$ is a finite set. This does not threaten our understanding of the notion of finitude. It simply means that first order logic lacks the power to express certain claims.

Given that the numerical formulas do not have any special connection to the concept of cardinality—they do not enshrine the very notion of cardinality—there would seem to be no good reason for approaching the issue of cardinality in the context of vagueness along the roundabout route via the truth values of numerical formulas. Simpler, and better, it seems, to define cardinality directly from the outputs of the counting process—for example in the ways that cardinality in the senses of $l$, $m$ and $e$ were defined in §7.3.

## 8 Conclusion

My concern in this paper has not been to add to the many existing proposals in the literature concerning the cardinality of vague collections, but to bring some order to the landscape—specifically, by bringing into focus the connection between the notions of counting, ordering and cardinality—a connection that is central in the classical case. I proposed a method for counting vague collections, and discussed the relationships between this method and various notions of ordering for vague sets. Turning then to the notion of cardinality, we saw that not all existing views concerning how we should answer the question as to how many things there are in a vague collection fit equally well with my proposal about how to count the members of such a collection. In particular, the idea that we should approach cardinality via certain formulas of a logical language—which has been quite influential in the recent philosophical literature—seems to me to be less attractive than other existing proposals.[18]

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