Vagueness, Uncertainty and Degrees of Belief: Two Kinds of Indeterminacy—One Kind of Credence

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Abstract

If we think, as Ramsey did, that a degree of belief that P is a stronger or weaker tendency to act as if P, then it is clear that not only uncertainty, but also vagueness, gives rise to degrees of belief. If I like hot coffee and do not know whether the coffee is hot or cold, I will have some tendency to reach for a cup; if I like hot coffee and know that the coffee is borderline hot, I will have some tendency to reach for a cup. Suppose that we take degrees of belief arising from uncertainty to obey the laws of probability and that we model vagueness using degrees of truth. We then encounter a problem: it does not look as though degrees of belief arising from vagueness should obey the laws of probability. One response would be to countenance two different sorts of degrees of belief: degrees of belief arising from uncertainty, which obey the laws of probability; and degrees of belief arising from vagueness, which obey a different set of laws. I argue, however, that if a degree of belief that P is a stronger or weaker tendency to act as if P, then this option is not open. Instead, I propose an account of the behaviour of degrees of belief that integrates subjective probabilities and degrees of truth. On this account, degrees of belief are expectations of degrees of truth. The account explains why degrees of belief behave in accordance with the laws of probability in cases involving only uncertainty, while also allowing degrees of belief to behave differently in cases involving only vagueness, and in mixed cases involving both uncertainty and vagueness. Justifications of the account are given both via Dutch books and in terms of epistemic accuracy.

1 Probabilities and Degrees of Truth

When the idea of degrees of truth—and the formalism of fuzzy set theory and logic—were new, there was apparently some confusion about the difference between probabilities and degrees of truth. Those days are behind us. Let S be the proposition that it is raining in Sydney at midday on 1 January 2012. If I tell you (say on the evening of 31 December 2011) that the *probability* of S is 0.7, then whether the probability involved is supposed to be objective or subjective, the idea is that at midday on 1 January 2012, either it will (clearly, definitely) be raining in Sydney, or it will (clearly, definitely) not be, but at the time at which I speak, either I simply do not know which it will be (if the probability is subjective) or it is objectively chancy which it will be (if the probability is objective). If, on the other hand, I tell you that the *degree of truth* of S is 0.7, then the idea is that at midday on 1 January 2012, it is neither clearly, definitely raining in Sydney, nor clearly, definitely not raining in Sydney—rather, the weather is in a state which is, say, borderline between a very heavy fog and a light rain.

I take it that this distinction between probabilities and degrees of truth is now well understood. Of course some say that we do not need degrees of truth—that we need only probabilities, perhaps only subjective probabilities, in order to give a good account of our world and of our place in it—but this claim is based on an understanding of the distinction between probabilities and degrees of truth.

Nevertheless, deep problems—both conceptual and technical—remain, concerning the relationships between probabilities and degrees of truth. For it is not as though the two notions can live separately: they come into contact and begin to interact—in puzzling ways—when we consider the notion of *degree of belief*. The aim of this paper is to resolve the puzzle and sort out the relationship.

2 Degrees of Belief

The notion of a *degree of belief*—more specifically, a degree of belief that S, or a degree of belief in the proposition that S—is a familiar one. When a term is familiar, it can be used without people asking 'What does that mean?'—and so it can be used in somewhat different senses by different authors, without these differences coming to light. So let me say that what I mean by a degree of belief that S is a strength of tendency to act as if S: one's degree of belief that S is a measure of the strength of one's tendency to act as if S. Here I follow Ramsey [1926, 65–6]:

> the degree of a belief is a causal property of it, which we can express vaguely as the extent to which we are prepared to act on it. ... it is not asserted that a belief is an idea which does actually lead to action, but one which would lead to action in suitable circumstances ... The difference [between believing more firmly and believing less firmly] seems to me to lie in how far we should act on these beliefs.

This is not meant to be a precise definition—it is a rough, guiding idea. The goal will be to make it more precise.

On terminology: I treat the terms 'degree of belief', 'partial belief' and 'credence' as synonyms. For reasons that we shall see, I do not treat the term 'subjective probability' as a synonym of 'degree of belief'.

Some might wish to start from a different guiding idea: that one's degrees of belief constitute one's internal picture of what the world is like—one's *epistemic state*; they might then object to the idea of cashing out degrees of belief in terms of tendencies to act. That approach is *not* in essential conflict with mine—except over the usage of the term 'degree of belief'. As we shall see, the view to be presented below makes a place for this idea of an agent's internal picture of what the world is like—her epistemic state—and does not try to cash out *this* notion in terms of action. I do not, however, associate the term 'degree of belief' *directly* with this epistemic state—and I *do* want to cash out degrees of belief (in the sense employed here) in terms of action. This will all become clear as we proceed: the point for now is just to make clear that in taking an agent's degree of belief that S to be a measure of the strength of her tendency to act as if S, I am not assuming that we can cash out the idea of the agent's epistemic state in terms of tendencies to act.

When I say that one's degree of belief that S is a measure of the strength of one's tendency to act as if S I do not mean that two persons who have the same degree of belief that S will behave in the same ways, or even have the same tendencies to behave in certain ways. The claim is that they will have the same tendency to act as if S. Whether a person's behaving in a certain way constitutes her acting as if S depends on her preferences (desires, utilities) and on her other beliefs.[1] For example, suppose that S is the proposition that there is a fragrant rose in Bob's garden. For a rose fancier, *approaching Bob's garden* might constitute acting as if S. For a person allergic to roses, or a rose fancier with false beliefs about the location of Bob's garden, *moving away from Bob's garden* might constitute acting as if S. So while two persons who have the same degree of belief that S will have

[1] Recall the Chisholm-Geach objection to behaviourism. See Chisholm [1957], Geach [1957].

the same tendency to act as if S—that is our guiding idea—in general they will behave in the same ways (described at the level of bodily movements, for example—rather than in terms of whether they are acting as if S) only if their other beliefs and desires are also the same: it is only then that *acting as if* S amounts to behaving in the same ways for both of them.

3 Uncertainty

One thing that weakens one's tendency to act as if S—that lowers one's degree of belief that S—is *uncertainty* regarding whether S is the case. For example, suppose that you need someone to help you reach an item on a high shelf. In this context, calling X and asking him or her for help is a way of acting as if X is tall. If you have heard that Bill is tall, but have not met him and are not sure whether this is true—or you have met him once, but cannot remember whether he is tall—then the strength of your tendency to call Bill will be reduced: the less likely you think it is that Bill is tall, the weaker your tendency to call him will be; the more likely you think it is that Bill is tall, the stronger your tendency to call him will be.

If uncertainty as to the truth of S were the only thing that affected one's degree of belief that S, then—for familiar reasons (e.g. Dutch Book arguments, accuracy arguments)[2]—it would be plausible to think that (rational) degrees of belief behave like probabilities, with degree of belief 0 in S indicating no tendency whatsoever to act as if S, and degree of belief 1 in S indicating the strongest possible tendency to act as if S. Hence we should arrive at *probabilism*: the view that degrees of belief are subjective probabilities.

4 Vagueness

However, the situation becomes more complex when we note that there is a second kind of indeterminacy—apart from uncertainty (i.e. *epistemic* indeterminacy)—that lowers one's degree of belief that S: *vagueness.* Consider again the situation in which you need someone to help you reach an item on a high shelf. In this context, asking X for help is a way of acting as if X is tall. Suppose, this time, that the potential helpers are in the room, in full view: you are not uncertain as to the height of any of them. If Bill is a borderline case of tallness, then the strength of your tendency to ask Bill for help will be reduced: the less clear (definite, determinate) a case of tallness

[2]On the former, see e.g. Ramsey [1926], de Finetti [1931, 1937]. On the latter, see e.g. Rosenkrantz [1981], van Fraassen [1983], Joyce [1998].

Bill is, the weaker your tendency to ask him will be; the more clear (definite, determinate) a case of tallness Bill is, the stronger your tendency to ask him will be.

Let's be a bit more careful. On the epistemic conception of vagueness,[3] the kind of indeterminacy involved here is just epistemic indeterminacy. Your reduced or strengthened tendency to ask Bill is simply a reflection of your greater or lesser uncertainty regarding whether Bill is tall. On the epistemic conception of vagueness, then, probabilism (the view that degrees of belief are subjective probabilities) is not threatened.

However, suppose we analyse vagueness in terms of degrees of truth (as I argue we should in Smith [2008]). More specifically, let us take these degrees of truth to be (modelled by) real numbers in the closed interval $[0, 1]$.[4] Now probabilism *is* threatened. For degrees of truth do not (in general) behave like probabilities,[5] and if greater or lesser degree of truth of S were the only thing that affected one's degree of belief that S, then it would be plausible to think that (rational) degrees of belief behave like degrees of truth, with degree of belief 0 in S indicating no tendency whatsoever to act as if S, and degree of belief 1 in S indicating the strongest possible tendency to act as if S.

5 Two Kinds of Degrees of Belief is One Kind Too Many

At this point, one might think of positing two kinds of degrees of belief: a kind related to uncertainty (where degrees of belief of this sort behave like probabilities) and a kind related to vagueness (where degrees of belief of this sort behave like degrees of truth).[6] However, given the guiding idea that one's degree of belief that S is a measure of the strength of one's tendency to act as if S, this will not do: there can be only one kind of degree of belief in this sense—for one cannot have two different strengths of tendency to act as if S (in a given set of circumstances).

[3]See e.g. Cargile [1969], Campbell [1974], Sorensen [1988, 2001], Williamson [1992, 1994], Horwich [1998].

[4]This brings us into the realm of fuzzy logics. At this stage, however, let us make no particular choice of fuzzy logic: no choice of particular operations on reals to associate with the connectives, no particular definition of logical consequence, and so on. So the discussion remains at a fairly abstract level, and goes through regardless of which particular choices we might make within the general realm of fuzzy logics.

[5]For example, the degree of truth of $A \wedge B$ is not, in general, the value that the probability calculus outputs for $A \wedge B$ when given as inputs the degrees of truth of A and B.

[6]Schiffer [2000] holds a view of this sort.

Consider, for example, the proposition that Fido is dangerous. When Fido enters the room, one will do some particular thing, for example sit still, or jump and run. When Fido looks at one, one will do some particular thing, for example tremble, or offer him some beef jerky. When Fido barks, one will do some particular thing, for example scream; and so on. One cannot both back away slowly *and* run screaming (at the same time), and it cannot both take Fido getting within two metres of one to make one run away, *and* require Fido getting within one metre to make one run. So one cannot both tend strongly to act is if Fido is dangerous *and* tend weakly to act as if Fido is dangerous.

Of course, there may be various factors contributing to one's tendency to act as if Fido is dangerous—and these factors may have different weights, and they may conflict with one another. For example, one might have been told by someone trustworthy that Fido is vicious, and yet one's own observations indicate that he is friendly. Nevertheless, the notion of 'strength of tendency to act as if S' is inherently a final, 'summing-up' notion: whatever complex factors go into *making* one's tendency to act as if Fido is dangerous have a certain strength, the tendency itself can have only one strength—not two different ones—just as, in any given situation, one will act in a certain way, and not in any other way. Given that a degree of belief *just is* a strength of tendency to act, this means that one cannot have two different degrees of belief in the same proposition.

Thus, while we should certainly like an account on which uncertainty and vagueness can both contribute to one's degrees of belief—an account on which uncertainty and vagueness can both be contributing factors in making one's tendencies to act have the strengths they have—we cannot accept an account according to which there are two distinct kinds of degrees of belief: the kind arising from uncertainty, and the kind arising from vagueness.

6 Design Brief

Summing up, we should like an univocal account of degrees of belief: one according to which there is only one kind of degree of belief. At the same time, these degrees of belief should be capable of different sorts of behaviours in different situations. In a context where the only kind of indeterminacy in play is epistemic—that is, where the only factor giving rise to intermediate degrees of belief is uncertainty—they should behave like probabilities. In a context where the only kind of indeterminacy in play is vagueness (understood in terms of degrees of truth), they should behave like degrees of truth. In contexts where both kinds of indeterminacy are in play at the same time, they might behave in other ways.

§7 presents an account of degrees of belief which fits this design brief.[7] §8 discusses the merits of this proposal and §9 situates it with respect to other views that have been presented in the literature.

7 Proposal: Degree of Belief is Expected Truth Value

The view to be presented here has three components:

1. the agent's epistemic state
2. the degrees of truth of propositions
3. the agent's degrees of belief in propositions.

Component 3 is (in a sense to be made clear) the resultant of components 1 and 2.

Component 1: Epistemic State. I shall take an agent's epistemic state to be represented by a probability measure over the space of possible worlds. The measure assigned to a set E of worlds indicates how likely the agent thinks it is that the actual world is one of the worlds in E. More formally, where W is the set of all possible worlds, the agent's epistemic state P is a function which assigns a real number between 0 and 1 inclusive to each subset of W, such that:

1. for every set $A \subseteq W$, $P(A) \geq 0$
2. $P(A \cup B) = P(A) + P(B)$ provided $A \cap B = \emptyset$
3. $P(W) = 1$.

Component 2: Degrees of Truth. Relative to each possible world, each proposition has a particular degree of truth. Thus, each proposition S is (associated with) a function $S : W \to [0, 1]$: the function that assigns to each world $w \in W$ the degree of truth of S at w. The relationships between (the functions associated with) various propositions are constrained in accordance with the laws of some fuzzy logic. That is, where $\oslash, \owedge, \ovee, \ominus$ are operations (of arities 2, 2, 2 and 1 respectively) on the fuzzy truth values (i.e. the reals in $[0, 1]$):

$$\begin{aligned}(S \to T)(w) &= S(w) \oslash T(w)\\ (S \wedge T)(w) &= S(w) \owedge T(w)\\ (S \vee T)(w) &= S(w) \ovee T(w)\\ (\neg S)(w) &= \ominus S(w)\end{aligned}$$

[7] The basic idea was presented in Smith [2008, §5.3] and Smith [2010]. The formulation here is more general.

In what follows, I assume only that the operations $\ominus, \oslash, \oslash$ and $\ominus$ agree with the classical operations on $\{0, 1\}$. Thus they might be the operations of Łukasiewicz logic, Product logic, Gödel logic, or any other fuzzy logic that agrees with classical logic as far as the truth values 0 and 1 are concerned.[8]

Component 3: Degrees of Belief. We have a measure over worlds (the agent's epistemic state P) and functions from worlds to real numbers (each proposition S). Thus each proposition S is a *random variable*. My proposal is that we identify the agent's *degree of belief* in S with her *expectation* (aka expected value) of S, written $E(S)$. Roughly, then, the agent's degree of belief in S is the weighted average of S's degrees of truth relative to all possible worlds—weighted according to how likely the agent thinks it is that each of those worlds is the actual world. (Worlds that the agent has ruled out—assigned zero probability—thus contribute no weight.)

Let's first work through this proposal in the finite case. Suppose that $w_1 \ldots w_n$ are all the possible worlds. The probability measure P is then fully determined by the assignments it makes to singletons:

$$P(\{w_i, \ldots, w_j\}) = P(\{w_i\}) + \ldots + P(\{w_j\})$$

So one assigns each world a degree of likelihood: a number indicating how likely one thinks it is that that world is the actual world. At the same time, each world w assigns each proposition S a degree of truth $S(w)$. One's degree of belief $E(S)$ in S is one's expectation for S, that is, one's expected value for S's degree of truth:

$$E(S) = P(\{w_1\}) \cdot S(w_1) + \ldots + P(\{w_n\}) \cdot S(w_n)$$

The picture generalises to the case of uncountably many possible worlds in a standard way. We suppose there to be a family $\mathcal{F}$ of measurable subsets of the space W of all possible worlds which is a σ-field, that is:

1. $W \in \mathcal{F}$
2. For all $A \in \mathcal{F}, \bar{A} \in \mathcal{F}$
3. For any countable number of sets $A_1, A_2, A_3 \ldots$ in $\mathcal{F}$, $\bigcup_i A_i \in \mathcal{F}$.

Only measurable propositions S—that is, propositions that are indeed random variables; that is, are such that for any real x, $\{w \in W : S(w) \leq x\} \in \mathcal{F}$—have expectations. So we have degrees of belief only in propositions S such that it makes sense to ask 'How likely do you take it to be that this

[8]For details of these logics, see e.g. Smith [2012].

proposition has a truth value within such-and-such limits?' Then, where W is the set of all possible worlds, w is a generic world in W, and $S = S(w)$ is a (measurable) function from worlds to degrees of truth (i.e. a proposition):

$$E(S) = \int_W S(w)dP$$

8 Merits of the Proposal

In the five subsections of this section, I discuss the following merits of the position presented in the previous section:

1. It fits with the guiding idea: one's degree of belief that S is a measure of the strength of one's tendency to act as if S.

2. It satisfies the design brief: it gives a univocal notion of degree of belief, but at the same time, allows that degrees of belief are capable of different sorts of behaviours in different situations.

   In particular, in a context where the only factor giving rise to intermediate degrees of belief is uncertainty, degrees of belief (i.e. expectations of degrees of truth) behave like probabilities, and in a context where the only kind of indeterminacy in play is vagueness (understood in terms of degrees of truth), they behave like degrees of truth.

3. It allows a familiar link between logical consequence and degrees of belief to be maintained.

4. Agents whose degrees of belief are expectations of truth value are immune to Dutch book.

5. If an agent's degrees of belief are expectations of truth value, then there is no other possible kind of belief state for that agent that is more accurate with respect to every possible world.

8.1 The Guiding Idea

In this section I argue that $E(S)$ is an accurate measure of one's tendency to act as if S. The argument is based on intuitions about an agent's strength of tendency to act as if S, in certain example scenarios: I shall argue that the values of $E(S)$ in these scenarios match the intuitive strengths of the agent's tendencies to act as if S. Before proceeding, recall the discussion at the end of §2: the claim is not that two persons who have the same degree of belief that S will behave in the same ways, or even have the same tendencies to behave in certain ways; the claim is that they will have the same tendency to act as if S.

Suppose that there are three 'open' worlds w_1, w_2 and w_3—that is, three worlds such that you are not certain that you are not in them. Let S be the proposition 'A tall person will win the race'. Suppose you don't know who will win, but you do know that it is either:

- the first man in a Sorites series leading from tall men to short men—this is the situation in w_1;
- the last man in the Sorites series—this is the situation in w_3; or
- a man right in the middle of the series—this is the situation in w_2.

You think that each of these three possibilities is equally likely:

$$P(\{w_1\}) = P(\{w_2\}) = P(\{w_3\}) = \frac{1}{3}$$

In w_1, S is 1 true; in w_2, S is 0.5 true; in w_3, S is 0 true. Hence:

$$E(S) = \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0.5 + \frac{1}{3} \cdot 0 = 0.5$$

Thus according to the view presented in this paper, your degree of belief in S is 0.5.

The claim being defended at present is that this is an accurate measure of your tendency to act as if S. Suppose you need a tall man for your basketball team, and you have a choice between signing up:

- Winner (i.e. the race winner, whoever it turns out to be);
- Biggy (whom you know to be of the same height as the first man in the Sorites series—hence 'Biggy is tall' is 1 true, and you know this, and so your expectation that Biggy is tall is 1);
- Shorty (whom you know to be of the same height as the last man in the Sorites series—so your expectation that Shorty is tall is 0); or
- Middleman (whom you know to be of the same height as the man in the middle of the Sorites series—so your expectation that Middleman is tall is 0.5).

You would sooner sign up Winner than Shorty, sooner sign up Biggy than Winner, and be indifferent between signing up Winner and Middleman. So your tendency to act as if x is tall does indeed mirror your expected truth value for 'x is tall'.

The foregoing assumes that your preference regarding team members is 'the taller the better'. But suppose your preference is for very tall players only. Then you would have no tendency to sign up Middleman, but some tendency to sign up Winner—even though your degrees of belief in 'Winner is tall' and 'Middleman is tall' would still be the same (i.e. 0.5). This is no problem for the view presented in this paper. The claim is that your degrees of belief in 'Winner is tall' and 'Middleman is tall' are the same—0.5—so you have the same tendency to act as if Winner is tall as to act as if Middleman is tall. This does not mean that you have the same tendency to sign them up. For if you want only very tall players, then signing up x does not constitute acting as if x is tall; rather, it constitutes acting as if x is very tall. Thus, the view presented in this paper correctly predicts that you will have no tendency to sign up Middleman (given your new preferences): because your degree of belief in 'Middleman is very tall' is 0, and signing up x is now acting as if x is very tall. The view also correctly predicts that you will have some tendency to sign up Winner—because your degree of belief in 'Winner is very tall' is:

$$\frac{1}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 0 = \frac{1}{3}$$

In sum: as your preferences change from 'the taller the better' to 'very tall', your degrees of belief in 'Middleman is tall', 'Middleman is very tall', 'Winner is tall' and 'Winner is very tall' remain the same (0.5, 0, 0.5 and 0.333). However, the significance of signing-up behaviour changes. At first, such behaviour constitutes acting as if the signed-up player is tall; with the new preferences, it constitutes acting as if the signed-up player is very tall. That is why two people who have the same degree of belief in 'Middleman is tall' might have different tendencies to sign up Middleman. My claim is that they will have the same tendency to act as if Middleman is tall. But if their preferences differ, then what counts as acting as if Middleman is tall for one person (e.g. signing up Middleman) might not count as acting as if Middleman is tall for the other person.

8.2 The Design Brief

This section shows that, on the view presented in this paper, degrees of belief (i.e. expectations of degree of truth) behave like probabilities in contexts where there is no vagueness in play, and like degrees of truth when there is no uncertainty in play. We begin by defining such contexts more precisely:

Definition 1 (vagueness-free situation)**.** An agent is in a vagueness-free situation (VFS) with respect to a proposition S iff there is a measure one set T of worlds (i.e. a set T such that $P(T) = 1$) such that $S(w) = 1$ or $S(w) = 0$ for every $w \in T$.

That is, the agent may not know for sure whether S is true or false, but she does absolutely rule out the possibility that S has an intermediate degree of truth: for she is certain that the actual world is somewhere in the set T, and everywhere in T, S is either 1 true or 0 true.

An agent is in a VFS with respect to a set Γ of propositions if she is in a VFS with respect to each of the propositions in Γ.

Definition 2 (uncertainty-free situation)**.** An agent is in an uncertainty-free situation (UFS) with respect to a proposition S iff there is a measure one set T of worlds and a $k \in [0, 1]$ such that $S(w) = k$ for every $w \in T$.

That is, it is totally ruled out that S has a degree of truth other than k: for the agent is certain that the actual world is somewhere in the set T, and everywhere in T, S is k true.

An agent is in a UFS with respect to a set Γ of propositions if she is in a UFS with respect to each of the propositions in Γ.

We can now establish that degrees of belief behave like probability assignments in vagueness-free situations (Propositions 1 and 3 below) and like degrees of truth in uncertainty-free situations (Propositions 2 and 4 below).[9] In situations that are neither vagueness-free nor uncertainty-free—that is, which involve uncertainty about the truth values of propositions, and where furthermore some propositions might, for all one knows, have intermediate degrees of truth—degrees of belief need not behave like probability assignments or like degrees of truth.

Proposition 1 (Degrees of belief equal probabilities in VFSs)**.** If an agent is in a VFS with respect to S, then

$$E(S) = P(\{w : S(w) = 1\})$$

Proposition 2 (Degrees of belief equal degrees of truth in UFSs)**.** If an agent is in a UFS with respect to S, then $E(S)$ equals the degree of truth that the agent is certain S has.

Proposition 3 (Degrees of belief behave like probabilities in VFSs)**.** Let Γ be a class of wffs, closed under the operations of forming wffs using the connectives $\rightarrow, \wedge, \vee$ and $\neg$, such that one is in a VFS with respect to Γ. Then one's degrees of belief of wffs in Γ behave like probabilities:

[9] For proofs of the four propositions, see Smith [2010]. The setting was less general in the earlier paper—but the proofs carry over, mutatis mutandis.

1. For all wffs $\gamma \in \Gamma$, $0 \leq E(\gamma) \leq 1$.

2. If $\gamma \in \Gamma$ is a tautology (i.e. never gets the value 0), then $E(\gamma) = 1$.

3. If $\gamma_1, \gamma_2 \in \Gamma$ are mutually exclusive (i.e. never both get the value 1), then $E(\gamma_1 \vee \gamma_2) = E(\gamma_1) + E(\gamma_2)$.

Proposition 4 (Degrees of belief behave like degrees of truth in UFSs)**.** Let Γ be a class of wffs, closed under the operations of forming wffs using the connectives $\rightarrow, \wedge, \vee$ and $\neg$, such that one is in a UFS with respect to Γ. Then one's degrees of belief of wffs in Γ behave like degrees of truth:

$$\begin{aligned} E(\gamma_1 \rightarrow \gamma_2) &= E(\gamma_1) \circledcirc E(\gamma_2) \\ E(\gamma_1 \wedge \gamma_2) &= E(\gamma_1) \owedge E(\gamma_2) \\ E(\gamma_1 \vee \gamma_2) &= E(\gamma_1) \ovee E(\gamma_2) \\ E(\neg\gamma) &= \ominus E(\gamma) \end{aligned}$$

8.3 Degrees of Belief and Logical Consequence

There is a familiar relationship between (rational) degrees of belief and logical consequence: in a valid argument, one's degree of belief in the conclusion cannot be less than one's degrees of belief in the premises. That is:

$$\text{if } \Gamma \models \alpha \text{ then } E(\alpha) \geq \bigwedge_{\gamma \in \Gamma} E(\gamma)$$

This will hold if we define consequence as preservation of minimum truth value of premisses—that is, the conclusion is never less true than the least-true premise:[10]

$$\Gamma \models \alpha \text{ iff on every model } \mathfrak{M}, [\alpha]_{\mathfrak{M}} \geq \bigwedge_{\gamma \in \Gamma} [\gamma]_{\mathfrak{M}}$$

where $[\alpha]_{\mathfrak{M}}$ is the degree of truth of α on the model $\mathfrak{M}$. For suppose $\Gamma \models \alpha$; then relative to every world, $[\alpha] \geq \bigwedge_{\gamma \in \Gamma} [\gamma]$; but then no matter what one's probability measure, $E(\alpha) \geq \bigwedge_{\gamma \in \Gamma} E(\gamma)$.

8.4 Dutch Books

Consider the following standard set-up; we assume for the moment that propositions can be only true or false. For each proposition θ, an agent is

[10] It need not hold if we define consequence in other ways.

to fix a betting quotient $B(\theta)$ which he regards as fair, in the sense that he will be willing to pay $s \times B(\theta)$ to enter into a bet on θ which pays him s if θ turns out to be true (and nothing if θ turns out to be false), whatever amount the stake s is: positive or negative.[11] Where $V(\theta)$ is the truth value of θ—1 if θ is true, 0 if θ is false—the agent's net return on such a bet can then be expressed as follows:[12]

$$s(V(\theta) - B(\theta))$$

If an agent enters into several bets, on propositions $\theta_1 \ldots \theta_m$, his net return on the whole set of bets is the sum of his net returns on each one (where s_i is the stake for the bet on θ_i, and $B(\theta_i)$ is the agent's betting quotient for θ_i):

$$\sum_{i=1}^{m} s_i(V(\theta_i) - B(\theta_i))$$

A Dutch Book against the agent is a set of bets that ensures the agent a negative net return, no matter what outcomes occur (i.e. no matter whether θ_i is true or false, for each θ_i):

$$\sum_{i=1}^{m} s_i(V(\theta_i) - B(\theta_i)) < 0$$

These ideas can all be transferred to the context of degrees of truth.[13] We simply suppose that a bet on θ at stake s pays $s \times V(\theta)$, where $V(\theta)$ may now be any real number in the interval $[0, 1]$. If θ has truth value 1 or 0, this reduces to the classical case considered above; but if θ has degree of truth (say) 0.7, then a bet on θ at stake (say) \$1 pays the agent \$0.7. The formula for the agent's net return on a set of bets is then just the same as above (except that this time, $V(\theta_i)$ may be any value in $[0, 1]$, not just 0 or 1, as in the classical case), as is the condition for a set of bets constituting a Dutch Book against the agent.

It is standard to suppose a close relationship between degrees of belief and fair betting quotients. In the present case, if an agent sets his betting

[11] It is the fact that s may be positive or negative that ensures that $B(\theta)$ is fair; cf. 'You cut, I choose' [de Finetti, 1974, 86, n.‡]. Note that if s is negative, then the agent 'paying' $s \times B(\theta)$ to enter into the bet amounts to his being paid $-s \times B(\theta)$ to enter into the bet, and the agent 'being paid' s amounts to his paying $-s$.

[12] If θ turns out to be true—$V(\theta) = 1$—the agent's net return is the winnings—s—minus the price he paid to enter the bet—$s \times B(\theta)$. If θ turns out to be false—$V(\theta) = 0$—the agent's net return is the 'winnings'—0—minus the price he paid to enter the bet—$s \times B(\theta)$.

[13] See for example Paris [2005] and Mundici [2006]—to mention just two important papers amongst the growing literature on this topic.

quotient for each θ_i to be his degree of belief that θ_i—his expectation of θ_i's degree of truth—then he will be immune to Dutch Book. This follows from a result of de Finetti [1974, 89].

Consider n-dimensional Euclidean space $\mathbb{R}^n$. A subset Γ of $\mathbb{R}^n$ is *convex* iff for every x and y in Γ and every λ in $[0, 1]$, $\lambda x + (1 - \lambda)y$ is in Γ (i.e. every point on the line segment joining x and y is in Γ). The *convex hull* of a set Γ is the smallest convex set containing Γ (i.e. the intersection of all convex sets that contain Γ). Now consider some propositions $\theta_1 \ldots \theta_m$, and the set of possible worlds W. We can think of these as together determining a set S of points in $\mathbb{R}^m$: where $[\theta]_w$ is the degree of truth of θ at world w, each world $w \in W$ determines the point $\langle[\theta_1]_w, \ldots, [\theta_m]_w\rangle$.[14] An agent's betting quotients for $\theta_1 \ldots \theta_m$ also determine a point B in $\mathbb{R}^m$: the point $\langle B(\theta_1), \ldots, B(\theta_m)\rangle$. De Finetti's result is that there is a Dutch Book against the agent (involving the propositions $\theta_1 \ldots \theta_m$) iff B lies outside the closed convex hull of S.

Suppose now that the agent sets his betting quotient for each θ_i to be his degree of belief that θ_i: his expectation of the degree of truth of θ_i. It follows that B is in the convex hull of S, and hence that the agent is immune to Dutch Book (with respect to the propositions $\theta_1 \ldots \theta_m$—but these are arbitrary). Perhaps the easiest way to think about this is as follows. We have a probability measure over the set W of worlds. We can derive from this a probability measure over the set S in $\mathbb{R}^m$ (S defined as in the previous paragraph). Recall that each point $s \in S$ arises from a world w; let f be the function that sends each world $w \in W$ to the corresponding point $s \in S$.[15] The measure of a subset of S will be the measure of the subset of W containing all worlds that are sent by f to a member of S. We can think of this process of generating a measure on S as applying *mass* to the points in S. The point B—representing the agent's degrees of belief in $\theta_1 \ldots \theta_m$—is then the *centre of mass* of the result. This centre of mass will always lie within the convex hull of S.[16]

The converse result also holds: the only way to avoid Dutch Book is to set one's betting quotients to be expectations of degrees of truth (relative to some underlying probability measure over W).[17] The key fact here is that every point in the closed convex hull of S is the centre of mass (or is a limit case) for some application of mass to the points in S [de Finetti, 1974, 90].

[14]It may be that two distinct worlds w and w' determine the same point in $\mathbb{R}^m$: this will happen if w and w' assign the same degrees of truth to the propositions $\theta_1 \ldots \theta_m$.

[15]f need not be one-one (cf. n.14).

[16]Cf. de Finetti [1974, 58].

[17]Of course, as far as this converse result is concerned, it need not be the probability measure that represents one's own epistemic state.

8.5 Accuracy

Ideally, an agent's degree of belief in each proposition θ would match θ's degree of truth (in the actual world). Of course, we are uncertain about the state of the actual world, so the goal of matching one's degree of belief in θ to θ's degree of truth is not a rule to live by. However, setting one's degree of belief in θ to one's expected value for θ's degree of truth *is* a rule one can live by—and it is guaranteed to lead to the ultimate goal of matching one's degree of belief in θ to θ's degree of truth, provided that sufficient evidence comes in. The process is as follows. As one discovers new evidence—which will, in general, be in the form 'S is n true', for some proposition S and some $n \in [0, 1]$—one conditionalises one's probability measure on the set S_n of worlds in which S is n true. That is, one updates one's old probability measure P to the new probability measure P' given by:[18]

$$P'(T) = P(T/S_n) = \frac{P(T \cap S_n)}{P(S_n)}$$

Provided sufficient evidence comes in, so that eventually one assigns measure one to the set containing just the actual world, one will end up with a degree of belief in each proposition θ that matches θ's degree of truth in the actual world (Proposition 2 above).

Not only does setting one's degrees of belief to expectations of truth value ensure a path (in principle) to degrees of belief in propositions that perfectly match their truth values: it furthermore provides a path that never strays further than necessary from the eventual goal—whatever the goal should turn out to be (i.e. whichever world is actual). That is, there is no other way of fixing one's degrees of belief (i.e. other than making them expectations of degrees of truth, relative to some underlying probability measure over worlds) that makes $B(\theta)$ closer to $[\theta]_w$ for every proposition θ and every world w. For if there were such a way, it would, in particular, render $B(\theta_i)$ closer to $[\theta_i]_w$ for every world w and every proposition θ_i in some finite set of propositions $\theta_1 \ldots \theta_m$—but as we saw in the previous section, the point B in $\mathbb{R}^m$ representing the agent's expected degrees of truth for arbitrary propositions $\theta_1 \ldots \theta_m$ must lie within the convex hull of the set S of points representing the possible degrees of truth of $\theta_1 \ldots \theta_m$—and the convex hull of S comprises precisely those points that cannot be moved in such a way as to reduce the (Euclidean) distance from all points in S [de Finetti, 1974, 90].

[18]This will not be defined if $P(S_n) = 0$. We must therefore assume that the agent does not start out with a probability measure that assigns measure zero to any set V of worlds such that she might subsequently get evidence that the actual world is in V.

9 Situating the Proposal in the Landscape of Theories

In the three subsections of this section, I situate the proposal of this paper with respect to other positions that have been advocated in the literature.

9.1 Contrast I: Probabilism

Consider the following quotations:

> Let our degrees of belief be represented by a probability measure, P, on a standard Borel space (Ω, F, P), where Ω is a set, F is a sigma-field of measurable subsets of Ω, and P is a probability measure on F. [Skyrms, 1984, 53]
>
> [By a reasonable initial credence function C] I meant, in part, that C was to be a probability distribution over (at least) the space whose points are possible worlds and whose regions (sets of worlds) are propositions. C is a non-negative, normalized, finitely additive measure defined on all propositions. [Lewis, 1986, 87–8]

These views equate degrees of belief (aka credences) with subjective probabilities. This view is known as *probabilism*:

> degree of belief = subjective probability

We can separate the probabilist equation into the following two equations:

1. epistemic state = probability measure
2. degree of belief = epistemic state

The view presented in this paper accepts 1, but rejects 2. (2 is rejected because intermediate degrees of belief may arise where there is no uncertainty, but there is vagueness.) So the view of this paper countenances the subjective probability measure—and agrees with the probabilists that it models the agent's epistemic state—but regards degrees of belief as resultants of this state and degrees of truth, rather than directly identifying degrees of belief with subjective probabilities.

Given bivalence, the difference just noted makes no difference (Propositions 1 and 3 above). However if we want to add degrees of truth to the mix, then the proposal of this paper (identifying degrees of belief with expectations of truth value) generalises smoothly, while probabilism (identifying degrees of belief with subjective probabilities) runs into the problem that vagueness (analysed in terms of degrees of truth) also gives rise to degrees of belief, but these degrees of belief do not behave like probabilities.

9.2 Contrast II: Belief Functions and Possibility Measures

As just noted, the view of this paper agrees with the probabilists in accepting equation 1, which identifies an agent's epistemic state with a probability measure. Some theorists reject this identification. In place of the probability measure, they substitute a function of the same sort (i.e. a function from sets of worlds to $[0, 1]$) but subject to weaker constraints:

- probability measure: additive

  $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \emptyset$

- Dempster-Shafer belief function:[19] super-additive

  $Bel(A \cup B) \geq Bel(A) + Bel(B)$ if $A \cap B = \emptyset$

- possibility measure:[20] maxitive

  $\Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}$

These theorists reject equation 1 because they think that our epistemic state is better modelled by a belief function or a possibility measure than by a probability measure; but they then typically accept equation 2—the aspect of probabilism rejected in this paper—which identifies an agent's degree of belief in a proposition with her epistemic state with respect to that proposition. However, one *could* reject 2, for my reasons, *and* reject 1, for their reasons: that is, replace the probability measure in the view presented in this paper with (for example) a belief function, and then follow through the rest of the development in §7 above, making the necessary flow-on changes.[21]

9.3 Contrast III: Field

Field [2000] supposes that an agent has a probability function P over propositions, and that the language includes a 'determinately' operator D. He proposes that the agent's degree of belief $Q(\alpha)$ in any proposition α is given by $Q(\alpha) = P(D\alpha)$. Thus one's degree of belief that α is one's subjective probability that determinately α.

One difference between the view of this paper and Field's view is as follows. Field says that "P should be thought of as simply a fictitious auxiliary used for obtaining Q" [p.16]; "P [should] not be taken seriously: except where it coincides with Q, it plays no role in describing the idealized agent" [p.19]. On the view of this paper, by contrast, subjective probabilities *do*

[19] See Dempster [1968], Shafer [1976].

[20] See Dubois and Prade [1988].

[21] See e.g. Smets [1981].

play an important role in describing an agent (they describe the agent's epistemic state)—even though they are not to be *identified* with degrees of belief (equation 2 of probabilism is rejected).

A second difference—one that poses a real problem for Field—is that degrees of belief in his sense are sensitive only to determinate truth (truth to degree 1, in the degree-theoretic setting), whereas degrees of belief in the sense of this paper are sensitive to all non-zero degrees of truth (provided that they have non-zero probability). Consider an example. Suppose that you see a leaf that is borderline red/orange. Your probability that it is determinately red (red to degree 1) is zero. Yet you have some tendency to act as if it is red: your degree of belief that it is red is not zero. Of course, if you need a perfectly red leaf, then you will have no tendency whatsoever to reach for the orangey-red one (even though your expectation that it is red is non-zero). But that is no problem for the view of this paper—because your expectation that the leaf is perfectly red (i.e. red to degree 1) *is* zero. On the other hand, if you need a red leaf, then you will have some tendency to reach for this one: less than for a perfectly red leaf, but more than for a green one.

10 Conclusion

The aim of this paper has been to shed light on the relationships between subjective probabilities, degrees of truth and degrees of belief. The basic picture presented here is as follows. Subjective probabilities provide a model of an agent's epistemic state. Degrees of truth feature in the analysis of vagueness. The two interact when it comes to degrees of belief—for one's degree of belief that S is a measure of the strength of one's tendency to act as if S, and this can be weakened not only by uncertainty as to whether or not S is the case, but also by vagueness in the extent to which S is the case. The positive proposal is that one's degree of belief in S is one's expectation of the degree of truth of S.[22]

[22]Versions of this paper were presented at a seminar at the Institute of Computer Science of the Academy of Sciences of the Czech Republic in Prague on 18 September 2009, at a Philosophy RSSS seminar at the Australian National University on 26 November 2009, at the Prague International Colloquium on Epistemic Aspects of Many-Valued Logic at the Institute of Philosophy of the Academy of Sciences of the Czech Republic on 15 September 2010, at the Metaphysical Indeterminacy Workshop II at the University of Leeds on 9 September 2011, at a joint session of the Australasian Association of Logic annual conference and the Twelfth Asian Logic Conference at Victoria University of Wellington on 15 December 2011, and at the Probability and Vagueness conference at the University of Tokyo on 21 March 2013; thanks to the audiences on those occasions for useful feedback. For helpful correspondence, I am grateful to Jeff Paris and Robbie Williams. Thanks to the two anonymous referees for their comments, and to the Australian Research Council

References

Richmond Campbell. The sorites paradox. *Philosophical Studies*, 26:175–191, 1974.

James Cargile. The sorites paradox, 1969. In Keefe and Smith [1997], pages 89–98.

Roderick M. Chisholm. *Perceiving: A Philosophical Study.* Cornell University Press, Ithaca, 1957.

Bruno de Finetti. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17:298–329, 1931.

Bruno de Finetti. La prévision: ses lois logiques, ses sources subjectives. *Annales De l'Institute Henri Poincaré*, 7:1–68, 1937.

Bruno de Finetti. *Theory of Probability: A Critical Introductory Treatment*, volume 1. John Wiley & Sons, Chichester, 1974.

Arthur P. Dempster. A generalization of Bayesian inference. *Journal of the Royal Statistical Society, Series B (Methodological)*, 30:205–47, 1968.

Didier Dubois and Henri Prade. *Possibility Theory: An Approach to Computerized Processing of Uncertainty.* Plenum, New York, 1988.

Hartry Field. Indeterminacy, degree of belief, and excluded middle. *Noûs*, 34:1–30, 2000.

Peter Geach. *Mental Acts: Their Content and Their Objects.* Routledge & Kegan Paul, London, 1957.

Paul Horwich. *Truth.* Clarendon Press, Oxford, second revised edition, 1998.

James M. Joyce. A nonpragmatic vindication of probabilism. *Philosophy of Science*, 65:575–603, 1998.

Rosanna Keefe and Peter Smith, editors. *Vagueness: A Reader.* MIT Press, Cambridge MA, 1997.

David Lewis. A subjectivist's guide to objective chance. In *Philosophical Papers*, volume II, pages 83–132. Oxford University Press, New York, 1986.

Daniele Mundici. Bookmaking over infinite-valued events. *International Journal of Approximate Reasoning*, 43:223–40, 2006.

J.B. Paris. A note on the dutch book method, 2005. Revised version of a paper in the Proceedings of the 2nd International Symposium on Imprecise Probabilities and their Applications, Ithaca, New York, 2001. Available at `http://www.maths.manchester.ac.uk/~jeff/papers/15.ps`.

F.P. Ramsey. Truth and probability, 1926. In Ramsey [1990], pages 52–94.

F.P. Ramsey. *Philosophical Papers.* Cambridge University Press, Cambridge, 1990. Edited by D.H. Mellor.

Roger D. Rosenkrantz. *Foundations and Applications of Inductive Probabil-*

for research support.

ity. Ridgeview, Atascadero CA, 1981.

Stephen Schiffer. Vagueness and partial belief. *Philosophical Issues*, 10:220–57, 2000.

Glenn Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, 1976.

Brian Skyrms. *Pragmatics and Empiricism*. Yale University Press, New Haven, 1984.

Philippe Smets. The degree of belief in a fuzzy event. *Information Sciences*, 25:1–19, 1981.

Nicholas J.J. Smith. *Vagueness and Degrees of Truth*. Oxford University Press, Oxford, 2008. Paperback 2013.

Nicholas J.J. Smith. Degree of belief is expected truth value. In Richard Dietz and Sebastiano Moruzzi, editors, *Cuts and Clouds: Vagueness, Its Nature, and Its Logic*, pages 491–506. Oxford University Press, Oxford, 2010.

Nicholas J.J. Smith. Many-valued logics. In Gillian Russell and Delia Graff Fara, editors, *The Routledge Companion to Philosophy of Language*, pages 636–51. Routledge, London, 2012.

Roy Sorensen. *Blindspots*. Clarendon Press, Oxford, 1988.

Roy Sorensen. *Vagueness and Contradiction*. Clarendon Press, Oxford, 2001.

Bas C. van Fraassen. Calibration: A frequency justification for personal probability. In R.S. Cohen and L. Laudan, editors, *Physics, Philosophy, and Psychoanalysis*, pages 295–319. D. Reidel, Dordrecht, 1983.

Timothy Williamson. Vagueness and ignorance, 1992. In Keefe and Smith [1997], pages 265–80.

Timothy Williamson. *Vagueness*. Routledge, London, 1994.