

CHANCES -- No. 17

by Richard Cowan

North
 ♠AK42
 ♥AK105
 ♦82
 ♣AJ6

North-South bid to 7NT with the cards shown here, using an auction which marks South with a 3334 shape and 17-18 HCP (including one ace and two kings).

South
 ♠J53
 ♥Q73
 ♦AKQ
 ♣KQ32

West leads the ♦J. South plays off winners to reach the 3-card ending shown in the next tableau, noting that both opponents followed suit to each trick.

He also noted that West played ♦10 and ♦9 on the second and third round of diamonds (as one would do if dealt ♦J109, ♦J109x or ♦J109xx -- using the principle of playing what one is known to hold). Also the ♠Q and ♥J did not fall.

North
 ♠4
 ♥K10
 ♦-
 ♣-

What is the chance, given the information to date, of making 7NT from the situation which we face after trick 10 – using the best line.

South
 ♠J
 ♥7
 ♦-
 ♣Q

This line obviously starts by cashing the ♣Q, pitching the ♠4 and observing both discards?

Then the ♥7 will be led.

If West shows out of hearts, or plays the ♥J, South's play is automatic. So the only decision comes when West plays ♥x on trick 12.

An incorrect argument in the case when West follows with ♥x on trick 12

It is tempting to argue solely on the basis of "vacancies". Midway through the 12th trick, at the point where South must decide which heart to play from dummy, he might reason as follows. West (having played ♥x) now has one unknown card, whilst East has two; therefore there is a 2:1 chance that the ♥J lies with East. Thus the ♥K is the best play.

This argument fails to take into account why West and East made their choices of discard on trick 11. Perhaps a certain choice is more likely to be made with some hands than others. So the question becomes: how are those 2:1 odds

modified by the knowledge that a defender chose in a certain way?

Kelsey and Glauert, in their book *Bridge Odds for Practical Players*, rather glibly apply the rule that a defender chooses from the safe choices available with equal weights. If there are two options, he chooses each 50% of the time (and independently on each occasion that he is confronted with that situation).

But is this not prejudging the situation? Is it clear that a West who has ♠x-♥x-♦x after trick 10 will choose equally often to pitch the ♠x or the ♥x? Or is the appropriate randomisation of his play 60:40 or 90:10? We don't really know at this stage.

The proper mathematical approach is not to prejudge; we let the proportion of times that he makes a given choice be a mathematical "unknown" and solve for this unknown.

The mathematical method

After trick 10, the six missing cards are ♠Q, ♠x, ♥J, ♥x and two ♦x. We know also, given that each player followed suit in the way described, that West was dealt one of the following hands:

♠xx-♥ <u>J</u> xx-♦JT9 <u>xx</u> -♣xxx	10/74
♠xx-♥ <u>xxx</u> -♦JT9 <u>xx</u> -♣xxx	10/74
♠ <u>Q</u> xx-♥xx-♦JT9 <u>xx</u> -♣xxx	10/74
♠ <u>xxx</u> -♥xx-♦JT9 <u>xx</u> -♣xxx	10/74
♠xx-♥ <u>Jx</u> xx-♦JT9 <u>x</u> -♣xxx	5/74
♠ <u>Qx</u> xx-♥xx-♦JT9 <u>x</u> -♣xxx	5/74
♠ <u>Q</u> xx-♥ <u>xxx</u> -♦JT9 <u>x</u> -♣xxx	5/74
♠ <u>Q</u> xx-♥ <u>J</u> xx-♦JT9 <u>x</u> -♣xxx	5/74
♠ <u>xxx</u> -♥ <u>xxx</u> -♦JT9 <u>x</u> -♣xxx	5/74
♠ <u>xxx</u> -♥ <u>J</u> xx-♦JT9 <u>x</u> -♣xxx	5/74
♠ <u>Q</u> xxx-♥ <u>J</u> xx-♦JT9-♣xxx	1/74
♠ <u>Q</u> xxx-♥ <u>xxx</u> -♦JT9-♣xxx	1/74
♠ <u>Q</u> xx-♥ <u>Jx</u> xx-♦JT9-♣xxx	1/74
♠ <u>xxx</u> -♥ <u>Jx</u> xx-♦JT9-♣xxx	1/74

The probability of each, given the information gathered during the first 10 tricks, is shown in the above list. Also noted in the list, in underlined bold font, are the three cards still held by West.

On the 1/74 case where West has ♠Q-♥Jx left, and the 10/74 case where West has ♠x-♦xx, one defender will be squeezed. South will recognise that the squeeze has been effective.

On four other cases, West will have the bare ♥J and automatically South makes 7NT. These cases have total weight 21/74. On two other cases of total weight 15/74, East will hold ♥Jx but not the ♠Q; on these, South is doomed to failure.

So we see that we can already “bank” 6 of the cases (having total chance 32/74) and are doomed on 2 cases whose total chance is 15/74.

With the remaining 6 cases where West will be able to play the ♥x on trick 12, there are chances of success for South and decisions he must make.

In these 6 cases, West may have (where the probability weight column “wt” must be divided by 74):

♠	♥	♦	wt	West would pitch
Qx	x	-	1	♠x
Q	x	x	5	♦x
x	Jx	-	1	♠x
x	x	x	5	♠x with prob s, ♦x with prob d
-	Jx	x	5	♦x
-	x	xx	10	♦x

Table I

We introduce s and d because we do not wish to rule out a randomised strategy. Naturally, s+d=1.

Now the corresponding hands for East are:

♠	♥	♦	wt	East pitches?
-	J	xx	1	♦x
x	J	x	5	♠x with prob S, ♦x with prob D
Q	-	xx	1	♦x
Q	J	x	5	♦x
Qx	-	x	5	♠x with prob p, ♦x with prob q
Qx	J	-	10	♠x

Table II

Here S+D=1 and p+q=1.

Event A: West pitches ♠x and East ♦x.

The cases extracted from Tables I and II which permit these discards are:

	♠	♥	♦	wt	The cards pitched
W	Qx	x	-	1	♠x from W
E	-	J	xx		♦x from E
W	x	Jx	-	1	♠x from W
E	Q	-	xx		♦x from E
W	x	x	x	5	♠x prob s from W
E	Q	J	x		♦x from E

Table III

So, given this situation,

$$\text{prob}\{E \text{ has } \heartsuit J \mid A\} = \frac{1+5s}{2+5s}$$

$$\text{prob}\{W \text{ has } \heartsuit J \mid A\} = \frac{1}{2+5s}$$

$$P\{A\} = (2+5s)/74.$$

Event B: West pitches ♦x and East ♠x.

The permitted cases extracted from Tables I and II are:

♠	♥	♦	wt	Pitches
Q	x	x	5	♦x from W
x	J	x		♠x from E with prob S
-	Jx	x	5	♦x from W
Qx	-	x		♠x from E with prob p
-	x	xx	10	♦x from W
Qx	J	-		♠x from E

Table IV

So, given this situation,

$$\text{prob}\{E \text{ has } \heartsuit J \mid B\} = \frac{10+5S}{10+5p+5S} = \frac{2+S}{2+p+S}$$

$$\text{prob}\{W \text{ has } \heartsuit J \mid B\} = \frac{p}{2+p+S}$$

$$\text{Also } P\{B\} = 5(2+p+S)/74.$$

Event C: West pitches ♦x and East ♦x.

The cases extracted from Tables I and II permitting these discards are:

♠	♥	♦	wt	Pitches
Q	x	x	5	♦x from W
x	J	x		♦x from E with prob D
x	x	x	5	♦x from W with prob d
Q	J	x		♦x from E
-	Jx	x	5	♦x from W
Qx	-	x		♦x from E with prob q

Table V

So, given this situation,

$$\text{prob}\{E \text{ has } \heartsuit J \mid C\} = \frac{d+D}{q+d+D}$$

$$\text{prob}\{W \text{ has } \heartsuit J \mid C\} = \frac{q}{q+d+D}$$

Here $P\{C\} = 5(q+d+D)/74$.

The overall chance of success

On trick 12, South leads his ♥7. When West plays ♥x, we must now consider what South does depending on the discards he has seen. To allow for the use of a randomised strategy, let us say that South plays the ♥K with probabilities a, b and c respectively after he sees the discarding events A, B or C, and the ♥10 with probabilities 1-a, 1-b and 1-c.

The overall probability P that South succeeds is

$$\begin{aligned}
 P &= 32/74 + \\
 &\quad P\{A\} [a \text{ prob}\{E \text{ has } \heartsuit J \mid A\} \\
 &\quad\quad + (1-a) \text{ prob}\{W \text{ has } \heartsuit J \mid A\}] + \\
 &\quad P\{B\} [b \text{ prob}\{E \text{ has } \heartsuit J \mid B\} \\
 &\quad\quad + (1-b) \text{ prob}\{W \text{ has } \heartsuit J \mid B\}] + \\
 &\quad P\{C\} [c \text{ prob}\{E \text{ has } \heartsuit J \mid C\} \\
 &\quad\quad + (1-c) \text{ prob}\{W \text{ has } \heartsuit J \mid C\}] \\
 \\
 &= 32/74 + 1/74 [a(1+5s) + (1-a)] + \\
 &\quad 5/74 [b(2+S) + (1-b)p] + \\
 &\quad 5/74 [c(d+D)+(1-c)q] \\
 \\
 &= [38+5as+5b(2-p+S)+5c(1+p-s-S)]/74 \quad (*)
 \end{aligned}$$

after replacing q, D and d by (1-p), (1-S) and (1-s) respectively. This can also be written as

$$P = [38+10b+5c + 5s(a-c) + 5(S-p)(b-c)]/74 \quad (**)$$

We have a mathematical game – South wishes to choose a, b and c to maximise P whilst East and West, acting in concert with perfect information (in this deal), choose p, s and S to minimise it.

Finding the max-min and min-max game-theoretic solution.

Finding South’s weights (a,b,c) is relatively easy. Look at (**). South recognises that his opponents can vary (S-p) between -1 and +1; they could make (S-p)=1 if South were to make (b-c)<0 or make (S-p)=-1 if (b-c)>0. In each case EW would gain. So South should nullify this by setting b=c.

Having done this he can maximise the rest by selecting b=c=1 and a=c=1 regardless of how EW select s. So he has maximised the worst reward he can get. Thus

$$\max (\min P) = 53/74 \quad \text{at } (a, b, c)=(1, 1, 1)$$

Another way to say this is that South has guaranteed that he cannot be exploited by his opponents to get a reward of < 53/74.

Here the choice of (a,b,c)=(1,1,1), which means that the ♥K is always played, is unique. Try any other combination, say (1, ½ , ½) or (0,1,1) and P will be < 53/74.

We now turn to (*) and EW’s selections. We note that the coefficients of a, b and c add to a constant, 15/74. If EW can keep all of these coefficients non-negative, then the worst outcome for EW is P=15/74. On the other hand, if one of the coefficients is negative, say the coefficient of b, South could place b=0 preventing EW getting a reward in this case. But with the two positive coefficients, he places a=c=1 and gains.

For example, suppose EW choose (s, S, p) = (½, 1, 0). Then $P = [38+15b+5(a-c)2]/74$ which South can exploit by choosing (1, 1, 0) to get $53/74 + 5/148$.

Intuitively one can see why. Knowing that p=0 means that South knows that East will always discard ♦x from ♠Qx-♦x. Thus in our analysis of event C,

$$\text{prob}\{E \text{ has } \heartsuit J \mid C\} = \frac{1/2}{1 + 1/2} = 1/3$$

$$\text{prob}\{W \text{ has } \heartsuit J \mid C\} = \frac{1}{1 + 1/2} = 2/3.$$

These comparisons have tipped in favour of South taking the finesse when both defenders pitch a ♦x.

So we have shown that

$$\min (\max P) = 53/74,$$

occurring provided EW keep all three coefficients of a, b and c in (*) non-negative.

This can be achieved by first selecting any value for p in the range $0 \leq p \leq 1$ and then choosing S and s such that

$$S + s \leq p + 1. \quad (\#)$$

So we have found the “von Neumann solution” to the game.

The outcome of $P = 53/74 = 71.62\%$ arises from South always playing the ♥K and EW choosing its discarding weights according to (#). This is a fairly liberal range of policies for EW, but notably does not allow all weightings.

The policy of $(s, S, p) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is allowed. This is the glib equally-likely discarding rule from cards that can affordably be pitched.

Note: This is the second version of my solution to appear on the internet. The earlier version contained an error on the distribution of West’s cards after 10 tricks. Thank you to Giampiero Bettinetti whose queries led to this revision.